Working Paper No. 115

Liberty, equality and impossibility: Some general results on the space of 'soft' preferences

by

S. Subramanian

Madras Institute of Development Studies
79, Second Main Road, Gandhi Nagar
Adyar, Chennai 600 020

July 1993

LIBERTY, EQUALITY, AND IMPOSSIBILITY: SOME GENERAL RESULTS ON THE SPACE OF 'SOFT' PREFERENCES

by
S. Subramanian

1. INTRODUCTION

In conventional social choice theory both individual and collective preferences are taken to be 'exact' or 'crisp'. In several recent works, many familiar impossibility results in collective choice theory have been generalized in a setting wherein the traditional choice framework has been relaxed to allow for vagueness in preference relations. 'Arrow type' problems of aggregation, and related issues, have been investigated by, among others, Barrett, Pattanaik and Salles (1986), Dutta (1987), and Barrett and Pattanaik (1990). A fuzzy version of a 'Sen-type' liberal paradox has been considered by Subramanian (1987).

With the exception of Barrett and Pattanaik (hereafter B-P, 1990), the concern in the remaining papers cited above has been with modelling vague preferences within a 'cardinal' framework of fuzzy sets. B-P (1990) undertake an 'ordinal' reformulation of 'cardinal' fuzzy sets, drawing on a version of Goguen's (1967) L-fuzzy set theory. The 'ordinalized' version of fuzzy sets is referred to by Basu, Deb and Pattanaik (1988) as 'soft' sets. In the present paper I examine, within a framework of 'soft' preferences, the comapatibility between Sen's (1970) principle of liberty and (a modified version of) Hammond's (1976) principle of equity. The modified equity principle is formulated in 'non-welfarist' terms, and - like the original version - exploits information on interpersonal utility comparisons. Despite this, and even in a relaxed framework of 'soft' preferences, the results on the possibility of 'liberal egalitarianism' are found to be essential discouraging.

This paper is organised as follows. Section 2 discusses basic concepts and definitions, and introduces some preliminaries relating to soft binary preference relations. Section 3 is concerned with the formulation of the equity and libertarian principles in a framework of soft preferences. The basic (relation - functional) results of the paper are set out in Section 4. In Section 5, the question of exact choice based on soft preferences is addressed; and in Section 6, the liberty-equity result is discussed in a choice functional setting. Concluding observations are offered in Section 7. Proofs of all results in the text have been relegated to an appendix at the end of the paper.

2. BASIC CONCEPTS

X ($\#X \geq 4$) $= \{x,y,z,\ldots\}$ is the finite set of all conceivable <u>social states</u> or <u>alternatives</u>, and N ($\#N \geq 2$) $= \{1,\ldots,i,\ldots,\#N\}$ is the finite set of individuals constituting society. For all $i \in N$, X_i ($\#X_i \geq 2$) is individual i's <u>personal issue</u>, i.e. a nonempty set of features of the possible social states that may be regarded as being personal to i (see Gibbard, 1974). (One could also include a set X_0 of 'public' features of the possible social states: this is an avoidable complexity which I shall, accordingly, ignore.) The set of all conceivable social states is given by the cartesian product of the X_i, viz, $X = \prod_{i \in N} X_i$. A typical social state $x \in X$ can be written as a list $x = (x_1,\ldots,x_i,\ldots,x_{\#N})$ where, for all $i \in N$, x_i is a description of person i's condition in the state-of-the-world x. A couple of definitions, which are of relevance to the formulation of the equity and liberty principles (to be discussed in Section 3), are now presented.

<u>Definition 2.1 (j-variants).</u> A pair of states $x,y \in X$ will be said to be <u>j-variants</u> if and only if $x_i = y_i$ $\forall i \in N \backslash \{j\}$ and $x_j \neq y_j$.

For all $j \in N$, let D_j be the set of all possible pairs of j - variants that can be constituted from X.

<u>Definition 2.2 (j,k - variants).</u> A pair of states $x,y \in X$ will be said to be <u>j,k - variants</u> if and only if $x_i = y_i$ $\forall i \in N \backslash \{j,k\}$ and $x_i \neq y_i$ $\forall i \in \{j,k\}$.

For all $j,k \in N$, let $D_{j,k}$ be the set of all possible pairs of j,k-variants that can be constituted from X.

I now introduce the notion of vague binary preference relations. In a 'cardinal' set theoretic framework, it is conventional to define a fuzzy binary preference relation as a function $P: X \times X \longrightarrow [0,1]$ such that, for all $x,y \in X$, $P(x,y)$ is interpreted as 'the degree of confidence' with which x is preferred to y, with $P(x,y)$ taking some value in the closed interval [0,1]. The specification of a precise degree of confidence with which any x is preferred to any y, on a cardinal scale going from zero to one, may be thought to do some injustice to precisely the notion of indeterminacy supposed to be captured by a fuzzy preference relation. An 'ordinal' reformulation of fuzziness would permit comparisons of levels of vagueness without any commitment to specification of precise degrees of vagueness on a cardinal scale. This serves as the motivation for the 'ordinally' reformulated 'soft' set-theoretic framework developed by Basu, Deb and Pattanaik (1988). In this paper, I shall rely on the 'soft' sets approach adopted by B-P (1990), which draws on a version of Goguen's (1967) L-fuzzy set theory. To this end, consider the following.

Let L ($\#L \geq 2$) be a finite ordered set $\{d_1,\ldots,d_m\}$, the elements of which are ranked by an exact antisymmetric ordering Q, whose asymmetric part is denoted by $\bar{Q}$, so that $d_m \bar{Q} d_{m-1} \bar{Q} \ldots \bar{Q} d_1$. A typical element $d \in L$ (see B-P, 1990) is to be construed as representing a 'degree of belonging' : thus, $\forall d,d' \in L$, $d\bar{Q}d'$ implies that the 'degree of belonging' as represented by d is greater than the 'degree of belonging' as represented by d'. d_m will be taken to represent 'belonging with complete confidence' and d_1 to represent 'not belonging with complete confidence'.

Definition 2.3 (Soft Binary Preference Relation). A soft binary preference relation (SBPR) P on X is a function $P: X \times X \longrightarrow L$.

Definition 2.4 (Exact or Crisp Binary Preference Relation). In definition 2.3, if $L = \{d_m, d_1\}$, then the function P is an exact or crisp binary preference relation on X.

I now introduce the notion of a soft extended preference relation, drawing on Arrow's (1967) notion of 'extended sympathy', which allows for ordinal interpersonal comparisons of utility whereby objects of the type 'being person j in state x' and 'being person k in state y' can be submitted to preference ranking.

Definition 2.5 (Soft Extended Binary Preference Relation). A soft extended binary preference relation (SEBPR) $\tilde{P}$ on the set XxN is a function $\tilde{P}$: $(X\times N)\times(X\times N)\rightarrow L$.

For all $i\in N$, P_i is individual i's SBPR on X_i and $\tilde{P}_i$ is individual i's SEBPR on XxN. In this paper, I shall employ the notation '$P_i(x,y)$' interchangeably with the notation '$\tilde{P}_i((x,i),(y,i))$'; in general, the latter notation will be preferred. Also, for all $j,k\in N$ and all $x,y\in X$, the statement '$\tilde{P}((x,j), (y,k))=d_m$' will also be written as '$(x,j)\tilde{P}(y,k)$' while, similarly, for all $x,y\in X$, the statement '$P(x,y)=d_m$' will be written equivalently as 'xPy'.

Let Σ be the set of all SBPRs on X. Let Σ_o be the set of all $P\in\Sigma$ satisfying (2.1) to (2.3) below:

(2.1) for all $x\in X$: $P(x,x) = d_1$ (irreflexivity);

(2.2) for all distinct $x,y\in X$: $P(x,y)=d_m\rightarrow P(y,x)=d_1$ (asymmetry); and

(2.3) for all distinct $x,y,z\in X$: $P(x,z)\,Q\,m(P(x,y),P(y,z))$

where
$$m(P(x,y), P(y,z)) = \begin{cases} P(x,y) & \text{if } P(y,z)QP(x,y); \\ P(y,z) & \text{if } P(x,y)QP(y,z) \end{cases}$$
(Maxmin transitivity of strict preference).

Let Σ_c be the set of all $P\in\Sigma$ such the P is crisp; and define the set $\Sigma_{oc} := \Sigma_o\cap\Sigma_c$.

Remark 2.6. In terms of the relationship of the set Σ_{oc} to traditional crisp preferences, it should be noted that elements of Σ_{oc} are what we would conventionally refer to as strict partial orderings (irreflexive, asymmetric and transtive binary relations).

Remark 2.7. There is no necessarily unique notion of transitivity of strict preferences in a fuzzy framework. In (2.3), I have presented the 'max-min' trasitivity rule widely employed in the literature (see B-P, 1990).

Next, I discuss a certain distinguished subset of the set of all strict soft extended binary preference relations, analogously with the preceding discussion of strict soft binary preference relations. Let $\tilde{\Sigma}$ be the set of all SEBPRs on XxN, and let $\tilde{\Sigma}_0$ be the set of all $\tilde{P} \in \tilde{\Sigma}$ satisfying (2.4) - (2.7) below:

(2.4) for all $x \in X$ and all $i \in N$: $\tilde{P}((x,i),(x,i))=d_1$ (irreflexivity);

(2.5) for all distinct $(x,j),(y,k) \in XxN$: $\tilde{P}((x,j),(y,k))=d_m \rightarrow \tilde{P}((y,k),(x,j))=d_1$ (asymmetry); and

(2.6) for all distinct $(x,j),(y,k),(z,l) \in XxN$: $\tilde{P}((x,j),(z,l))Qm(\tilde{P}((x,j),(y,k)), \tilde{P}((y,k),(z,l)))$, where $m(\tilde{P}((x,j),(y,k)), \tilde{P}((y,k),(z,l)))=\tilde{P}((x,j),(y,k))$ if $\tilde{P}((y,k),(z,l))Q\tilde{P}((x,j),(y,k))$; $=\tilde{P}((y,k),(z,l))$ if $\tilde{P}((x,j),(y,k))Q\tilde{P}((y,k),(z,l))$ (max-min transitivity of strict preference).

Let $\tilde{\Sigma}_C$ be the set of all $\tilde{P} \in \tilde{\Sigma}$ such that $\tilde{P}$ is crisp; and define the set of all crisp extended preference relations on X which are transitive by $\tilde{\Sigma}_{OC} := \tilde{\Sigma}_O \cap \tilde{\Sigma}_C$.

Finally, I consider the question of aggregating individual soft preferences into a collective soft preference. It will be assumed that society's problem is to arrive at a collective soft ranking of alternative social states on the basis of information provided by the set of individual soft extended preference relations on the set XxN.

Definition 2.8 (Generalized Soft Aggregation Rule): A generalized soft aggregation rule (GSAR) is a function $f:(\tilde{\Sigma}')^{\#N} \rightarrow \hat{\Sigma}$ where $\emptyset \neq \tilde{\Sigma}' \subseteq \tilde{\Sigma}$ and $\emptyset \neq \hat{\Sigma} \subseteq \tilde{\Sigma}$, such that, for every #N-tuple of individual soft extended preference relations $(\tilde{P}_i)$ in its domain, the aggregation rule f specifies a unique soft preference ranking of X.

Remark 2.9. There are at least two reasons why it may be of interest to consider rules that take values in soft (social) preferences. The first is that there may be some intrinsic merit to allowing for the possibility that collective judgements are, from the point of view of descriptive realism and normative reasonableness, fuzzy and imprecise in a way which traditional aggregation theory, with its emphasis on preferences that are unvaryingly undivided over pairs of alternatives, denies. The second reason is an essentially technical one: given an aggregation rule with domain D and range R, a theorem to the effect that there exists no rule $f: D \rightarrow R$ satisfying a specified set of properties $c_1, \ldots, c_k$ is a weaker result than a theorem that asserts that there exists no rule $f: D \rightarrow R'$ satisfying $c_1, \ldots, c_k$ when $R' \supseteq R$; since the set of soft social preferences is a superset of the set of exact social preferences, there is some mileage - in terms of greater generality - to be gained from proving an impossibility theorem in a framework of soft rather that exact social preferences.

We may wish to restrict the GSAR f by requiring it to satisfy what we may regard as desirable properties in an aggregation mechanism. In particular, society may be interested in constraining the collective choice mechanism to satisfy the ethical principles of equity and liberty. Whether this is a reasonable expectation will be investigated in the rest of this paper. But first, a formulation of the equity and the liberty principles.

3. THE EQUITY AND LIBERTARIAN PRINCIPLES

3.1 The Crisp Framework

Hammond (1976) provides a social choice theoretic approximation of Sen's (1973) weak equity axiom which is essentially a rule for an optimal distribution of income between two individuals, given the incomes of all other individuals. The weak equity axiom demands that in distributing a given income between two individuals, a larger share should go to the more disadvantaged individual. Hammond translates this requirement, in the social choice framework, in terms of the following principles of 'equity' and 'weak equity'. (In the definitions provided below, for all i, I_i is the symmetric component - representing 'indifference' - of the 'weak' preference ordering R_i; $\tilde{P}$ is the strict extended preference ordering of an hypothetical 'ethical observer'; R is the 'weak' social preference ordering; and P is the asymmetric factor of R.)

Definition 3.1.1 (Equity): Equity demands that for all $x,y \in X$, and for all $j,k \in N$, if xP_jy, yP_kx, $[\forall_i \in N \backslash \{j,k\}: xI_iy]$, and $(x,k)\tilde{P}(x,j)$ and $(y,k)\tilde{P}(y,j)$, then xPy.

Definition 3.1.2 (Weak Equity). The Weak Equity principle is derived from the Equity principle by replacing 'xPy' in Definition 3.1.1 by 'xRy'.

Hammond (1976, p.795) explains his formulation of the Equity and Weak Equity Principles, based on Sen's Weak Equity Axiom, as follows (in the ensuing quotation, I have resorted to some very minor notational changes from the original):

> Suppose that y denotes an equal distribution of income, and x an alternative distribution in which j's income has risen and k's income has fallen, with all other incomes remaining the same [emphasis added]. Then Sen's axiom effectively requires that, if x is close enough to y, then x is socially preferred to y. This is assuming that j has a lower level of welfare than k in state y, i.e. that j enjoys fewer advantages than k.
> One might now try to extend this principle to social choices which are more general than choices of income distribution. Then, the obvious and essential features of the social choice examined in the previous paragraph are as follows:
>
> (i) xP_jy, yP_kx, and for all $i \notin \{j,k\}, xI_iy$; (ii) $(y,k)\tilde{P}(y,j)$; and (iii) x is close to y. Only condition (iii) is imprecise. There are a number of ways one might try to make it precise. One way is to insist that x must re-distribute income so that j, who enjoyed

> fewer advantages than k in state y still enjoys fewer advantages than k in state x, i.e (iii) $(x,k)\tilde{P}(x,j)$.But in fact, one can weaken this slightly to require that if conditions (i), (ii) and (iii) are satisfied, then x should be weakly preferred to y by society. [This is the Weak Equity Principle of Definition 3.1.2].

The difficulty with Hammond's translation of Sen resides in the italicized part of the sentence in the first paragraph of the above quotation. In applying the equity principle in order to decide on whether to respect j's or k's preference over the pair of states (x,y), the requirement that the personal features of all individuals other than j and k be the same in both states x and y seems to have been imperfectly captured by Hammond's requirement that all individuals other than j and k be *indifferent* as between x and y. While individuals may (from a normative point of view) be expected to be indifferent between states which are invariant with respect to their personal features, it is not clear that they will, in practice, be so indifferent. In translating Sen's Weak Equity Axiom into its social choice counterpart, there does seem to be a case for imposing more structure on the pair of social states under comparison - by requiring, typically, that *x and y be a pair of j,k-varients* (for a definition of which see Section 2.) There also appears to be a case for dropping Hammond's 'indifference' requirement: the preferences of individuals other than j and k should simply not be relevant for determining the social preference over the pair of states (x,y) (note that this stricture does not apply to their *extended* preferences). In line with this reasoning, I advance the following *modified* Equity and Weak Equity principles:

Definition 3.1.3 (Modified Equity). *Modified Equity* demands that for all $x,y \in X$ and for all $j,k \in N$, if $(x,y) \in D_{j,k}$, $xP_j y$, $yP_k x$ and $[\forall i \in N: (x,k)\tilde{P}_i(x,j)\ \&\ (y,k)\tilde{P}_i(y,j)]$, then xPy.

Definition 3.1.4. (Modified Weak Equity). *Modified Weak Equity* is derived from Modified Equity by replacing 'xPy' in Definition 3.1.3 by 'xRy'.

Remark 3.1.5. Note that in the modified versions of the equity principles, the fact of j being the more disadvantaged individual is represented by the requirement that every individual prefers being person k to being person j in both the states x and y: this is one way of basing social preference on individual extended preference orderings, rather than on a single extended preference ordering purporting to be that of an impartial 'ethical observer'.

Remark 3.1.6. Notice also that Hammond's formulation of the equity principle is such as to make it satisfy Arrow's (1963) condition of 'neutrality', which essentially requires that if information on individual utilities with respect to any pair of states (x,y) is identical to information on individual utilities with respect to any other pair of states (w,z), then the social ranking of the pair (x,y) should be identical to the social ranking of the pair (w,z). (This property Sen (1979) also calls 'welfarism'). However, because of the structure imposed on pairs of alternatives to qualify them for comparison in terms of the Modified Equity Axiom the latter violates the neutrality axiom: Modified Equity is a non-welfarist principle (that is, a principle which is sensitive to considerations other than solely the utility content of alternative states-of the-world.)

Next, the libertarian principle. Sen's (1970) principle of minimal liberty requires that each of at least two individuals in society sould be decisive over at least one pair of alternatives each, with the alternatives in each pair differing only in a feature of personal relevance to the concerned individual. Formally, we have:

Definition 3.1.7 (Minimal Liberalism). Minmal Liberalism requires that $\exists j,k \in N$ and $\exists[(x,y)\in D_j,(w,z)\in D_k]$ such that if xP_jy (respectively, yP_jx), then xPy (respectively, yPx), and if wP_kz (respectively, zP_kw), then wPz (respectively, zPw).

Precisely in the spirit of the weakened version of the equity principle, one can have a weakened version of the minimal liberty principle which allows only for a weak veto on assigned pairs of alternatives:

Definition 3.1.8 (Weak Minimal Liberalism). Weak Minimal Liberalism requries that $\exists j,k\in N$ and $\exists[(x,y)\in D_j,(w,z)\in D_k]$ such that if xP_jy (respectively, yP_jx), then xRy (respectively, yRx), and if wP_kz (respectively, zP_kw), then wRz (respectively, zRw).

3.2 The Soft Framework

The following definitions are straightforward translations, in a framework of soft preferences, of the corresponding definitions in the crisp framework:

Definition 3.2.1 (Modified Equity): A GSAR satisfies Modified Equity (ME) if and only if for all $x,y\in X$, for all $j,k\in N$, and all $(\tilde{P}_i)_{i\in N}$ in the domain of the GSAR:

if $(x,y)\in D_{j,k}$, $\tilde{P}_j((x,j),(y,j))=d_m$ & $\tilde{P}_j((y,j),(x,j))=d_1$,

$\tilde{P}_k((y,k),(x,k))=d_m$ & $\tilde{P}_k((x,k),(y,k))=d_1$, and $\forall i\in N$:

$\tilde{P}_i((x,k),(x,j))=\tilde{P}_i((y,k),(y,j))=d_m$ & $\tilde{P}_i((x,j),(x,k))=$

$\tilde{P}_i((y,j),(y,k))=d_1$, then $P(x,y)\bar{Q}\ P(y,x)=d_1$.

Definition 3.2.2 (Modified Weak Equity). Modified Weak Equity (MWE) is derived from Modified Equity by replacing '$P(x,y)\bar{Q}\ P(y,x)=d_1$' in Definition 3.2.1 by '$P(x,y)Q\ P(y,x)=d_1$'.

Definition 3.2.3 (Minimal Liberalism): A GSAR satisfies Minimal Liberalism (ML) if and only if there exist at least two distinct individuals j and k and two distinct doubletons of alternatives $(x,y)\in D_j$ and $(w,z)\in D_k$ such that for all $(\tilde{P}_i)_{i\in N}$ in the domain of the GSAR:

(3.2.1) $[(\tilde{P}_j((x,j),(y,j))=d_m$ & $\tilde{P}_j((y,j),(x,j))=d_1]$

(resp., $[\tilde{P}_j((y,j),(x,j))=d_m$ & $\tilde{P}_j((x,j),(y,j))=d_1])$

$\longrightarrow$ $[P(x,y)\bar{Q}\ P(y,x))=d_1]$ (resp., $[P(y,x)\bar{Q}\ P(x,y)=d_1]$); and

(3.2.2) $[\tilde{P}_k((w,k),(z,k))=d_m \ \& \ \tilde{P}_k((z,k),(w,k))=d_1]$

(resp., $[\tilde{P}_k((z,k),(w,k))=d_m \ \& \ \tilde{P}_k((w,k),(z,k))=d_1]$)

$\longrightarrow$ $[P(w,z)\bar{Q}P(z,w)=d_1]$ (resp., $[P(z,w)\bar{Q}\ P(w,z)=d_1]$).

Definition 3.2.4 (Weak Minimal Liberalism).

Weak Minimal Liberalism (WML) is obtained from ML by replacing '$[P(x,y)\bar{Q}P(y,x)=d_1]$ (resp., $[P(y,x)Q\bar{P}(x,y)=d_1]$)' in (3.2.1) by '$[P(x,y)Q\ P(y,x)]=d_1]$ (resp., $[P(y,x)Q\ P(x,y)=d_1]$)' and '$[P(w,z)\bar{Q}\ P(z,w)\ =d_1]$ (resp., $[P(z,w)\bar{Q}\ P(w,z)=d_1]$)' in (3.2.2) by '$[P(w,z)Q\ P(z,w)=d_1]$ (resp., $[P(z,w)Q\ P(w,z)=d_1]$)'.

Remark 3.2.5: All references to the equity and liberty principles will be with respect to the generalized 'soft' definitions 3.2.1 - 3.2.4. Given the preceeding inventory of concepts and definitions, the main (relation-functional) results of the paper can now be stated.

4. ON THE POSSIBILITY OF EGALITARIAN LIBERALISM

The following proposition is true.

Theorem 4.1 There exists no GSAR $f:(\tilde{\Sigma}_{OC})^{\#N} \to \Sigma_{OC}$ satisfying conditons ML and ME if X contains a subset $A=\{x,y,w,z\}$ such that $(x,y)\in D_j$ and $(w,z)\in D_k$ for some $j,k\in N$, and either

(i) $\#(\{x,y\}\cap\{w,z\})=1$

or

(ii) $\{x,y\}\cap\{w,z\}=(\{x,y\}\times\{w,z\})\cap D_j = (\{x,y\}\times\{w,z\})\cap D_k = \emptyset$ and $(\{x,y\}\times\{w,z\})\cap D_{j,k} \neq \emptyset$.

Remark 4.2. A proof of the theorem is omitted since Theorem 4.1, which is in the crisp framework, is implied by and subsumed under its soft generalization, Theorem 4.4, which will be presently stated.

To fix ideas, it may be useful to furnish an example illustrating the intuition underlying Theorem 4.1. I present below an example which takes some liberties with a Wodehousean situation.

Example 4.3: Police Constable Oates would give anything to ensure that the dog Bartholomew does not accompany his mistress Stephanie Byng on her walks, for there is between man and dog a deep-seated enmity. If Bartholomew makes unfriendly noises at Oates it is, in Miss Byng's view, because Oates is aggravating the dog by doing his beat on a bicycle when the right thing to do would be for him to walk which, according to Miss Byng (although Oates does not regard this as being the 'point at tissue'), would also help in knocking off some of the policeman's fat.

We can now define the following social states. x is a state in which Miss Byng is accompanied on her walks by her dog, and Oates does his beat on his bicycle; y is identical to x in all respects save that in y Oates does his beat on foot; and z is identical to y in all respects save that in z Miss Byng leaves her dog behind at home.

Clearly, the states x,y are a pair of 'P.C Oates-variants', and the states y,z a pair of 'Miss Byng-variants'. Since other things equal the policeman prefers travelling by bicylce to travelling on foot, and the lady prefers walking with Bartholomew to walking without him, condition ML assures us that xPy and yPz so that, thanks to transitivity of the strict social preference relation, we have xPz.

Assume further (i) that Miss Byng prefers state x to state z (her overriding concern is that the dog Bartholomew should accompany her on her walks); (ii) that Oates prefers state z to state x (never mind if he has to do his beat on foot, so long as he does not have to have his day blighted by an encounter with that dog); and (iii) that in everyone's view Oates' welfare level (possibily by virtue of his being a policeman which, according to Wodehouse, makes for a generally jaundiced view of life) is lower than Miss Byng's in both states x and z. Given this, and noting that the pair of states (x,z) is a pair of 'Byng, Oates-varients', condition ME will dictate zPx. But this contradicts xPz derived earlier. Sir Watkyn Basset, Justice of the Peace for Totleigh-on-the-Wold, should be pardoned if under the circumstances he finds himself unable to find a satisfactory solution to the problem - one which serves the interests of both equity and individual liberty.

The following result, in which the range of the aggregation rule is expanded to admit soft preferences, is a generalization of the 'crisp' result embodied in Theorem 4.1.

<u>Theorem 4.4.</u> There exists no GSAR $f:(\tilde{\Sigma}_{OC})^{\#N} \longrightarrow \Sigma_O$ satisfying conditions ML and ME if X contains a subset $A=\{x,y,w,z\}$ such that $(x,y)\in D_j$ and $(w,z)\in D_k$ for some $j,k\in N$, and either

(i) $\#(\{x,y\}\cap\{w,z\}) = 1$

or

(ii) $\{x,y\}\cap\{w,z\} = (\{x,y\}\times\{w,z\})\cap D_j = (\{x,y\}\times\{w,z\})\cap D_k = \emptyset$ and $(\{x,y\}\times\{w,z\})\cap D_{j,k} \neq \emptyset$.

Proof. See Appendix.

One can now obtain a possibility result if the liberty and equity principles are diluted to their respective weakened versions; indeed, now the domain of the aggregation rule can also be expanded to accomodate #N - tuples of 'properly' soft extended preference relations. The following proposition is true.

Theorem 4.5. There exists a GSAR $f:(\tilde{\Sigma}_0)^{\#N} \rightarrow \Sigma_0$ satisfying conditions WML and MWE.

Proof. See Appendix.

Remark 4.6. While the existence result of theorem 4.5 is certainly more encouraging than the impossibility result of Theorem 4.4, the full implication of the result - in terms of the possibility of consistent social choice - becomes transparent only when we move from a 'relation-functional' to a 'choice-functional' framework of preference aggregation. It seems reasonable to believe - see, for example, Dutta (1987) - that while preferences can be vague choice, of necessity, must be exact (after all, one chooses or one does not choose: it is hard to confer any sensible interpretation on the notion of 'degrees' of choice); and it is the question of choice functions and exact choice that is addressed in the following section.

5. CHOICE FUNCTIONS AND EXACT CHOICE

There is an extensive literature on the theory of exact choice in a framework of exact preferences; standard references would include Arrow (1959), Richter (1966), and Sen (1971). Considerable recent work is also available on exact choice in a framework of fuzzy preferences; of particular interest are the papers by Basu (1984), Dutta, Panda and Pattanaik (1986), Barrett, Pattanaik and Salles (1990), Dutta (1987), and Basu, Deb and Pattanaik (1988, especially section 6.)

I begin with a definition of an exact choice function. In what follows, $\mathfrak{X}$ will stand for power set of X, the set of all conceivable alternatives.

Definition 5.1. (Crisp Choice Function): A Crisp Choice Function (CCF) is a mapping $C:\mathfrak{X}\to\mathfrak{X}$ such that for all $A(\neq\emptyset)\in\mathfrak{X}$, $\emptyset\neq C(A)\subseteq A$.

Remark 5.2. Definition 5.1 tells us that a (crisp) choice function is a function C which, for every nonempty set A in the power set of X assigns a nonempty subset of A, C(A), which is the set of chosen elements from A, or the choice set of A.

The issue of rationalizability of a choice function revolves around the investigation of a systematic link between the choice set of A and a binary preference relation, in terms of which the choices made from A can be 'explained'. One particular notion of rationalizability of a CCF by a soft binary preference relation - that of H-rationalizability (see Dutta, Panda and Pattanaik, 1986, and Dutta, 1987) - is discussed below.

Let $A\in\mathfrak{X}$, and let P be any SBPR. For every $a\in A$, let $a^*_A\in A$ be defined such that $P(a,b)QP(a,a^*_A)$ $\forall b\in A\setminus\{a\}$. In some straightforward sense, a^*_A can be interpreted as a's 'closet rival' in A, in terms of the preference relation P. Now define the set $B(A,P):=\{x\in A|P(x,x^*_A)QP(y,y^*_A)\quad\forall y\in A\}$. Stated informally, an alternative x will belong to the set B(A,P) - which it is convenient to interpret as the set of 'P-best elements of A'- if and only if the extent to which x is preferred to its 'nearest rival' is at least as much as the extent to which any other alternative y in A is preferred to its (y's) 'nearest rival'. The notion of H-rationalizability can now be precisely defined.

Definition 5.3. (H-Rationalizability). A CCF C is H-rationalizable by a SBPR P if and only if for all $A\in\mathfrak{X}$, $C(A)=B(A,P)$.
(That is, C is H-rationalizable by P if the choice set of A consists of those elements which are P-best in A).

One of a number of 'consistency properties' of choice functions, intensively investigated in the literature by, among others, Sen (1977) and Bordes (1976), is a strong 'expansion' consistency property called β^+, which is now defined.

Definition 5.4. (Expansion Consistency: Property β^+). Property β^+ requires of a CCF that for all $x,y \in X$ and all $A,B \in \mathfrak{X}$ such that $A \subseteq B$: $[x \in C(A) \& y \in A] \rightarrow [y \in C(B) \rightarrow x \in C(B)]$.

(That is, if x belongs to the choice set of a set A which contains y, then y cannot belong to the choice set of B which is an expansion of A without x also belonging to the choice set of B).

A result concerning Property β^+ which is of importance for future use, is presented below. This result is a close parallel of one stated and proved, in a 'cardinal' framework of fuzzy preferences, in Dutta, Panda and Pattanaik (1987; see their proposition 5.4(a)).

Lemma 5.5. Let C be a CCF. If C is H-rationalizable by a SBPR $P \in \Sigma_0$, then C satisfies Property β^+.

Proof. See Appendix.

Lemma 5.5 has an important role to play in the choice functional approach to aggregating individual preferences, to which I now turn.

Definition 5.7 (Crisp Collective Choice Function). A crisp collective choice function (CCCF) is a mapping $C: \mathfrak{X} \times (\tilde{\Sigma}')^{\#N} \rightarrow \mathfrak{X}$ (where $\emptyset \neq \tilde{\Sigma}' \subseteq \tilde{\Sigma}$) such that, for all $A(\neq \emptyset) \in \mathfrak{X}$ and for all $(\tilde{P}_i)_{i \in N} \in (\tilde{\Sigma}')^{\#N}$, $\emptyset \neq C(A, (\tilde{P}_i)_{i \in N}) \subseteq A$.

The notion of H-generation of a crisp collective choice function by a generalized soft aggregation rule (see Dutta, 1987) is now considered.

<u>Definition 5.8 (H-generation of a CCCF by a GSAR):</u> Let $f:(\tilde{\Sigma}')^{\#N} \to \hat{\Sigma}$ ($\emptyset \neq \tilde{\Sigma}' \subseteq \tilde{\Sigma}$ and $\emptyset \neq \hat{\Sigma} \subseteq \Sigma$) be a GSAR, and let $C:\mathcal{X}\times(\tilde{\Sigma}')^{\#N} \to \mathcal{X}$ be a CCCF. Then C will be said to <u>be H-generated by f</u> if and only if for all $A \in \mathcal{X}$ and all $(\tilde{P}_i)_{i\in N} \in (\tilde{\Sigma}')^{\#N}$, C is H-rationalizable by f, that is, $C(A,(\tilde{P}_i)_{i\in N}) = B(A,f((\tilde{P}_i)_{i\in N}))$, where $B(A,f((\tilde{P}_i)_{i\in N}))$ is the set of 'f-best' elements in A.

The choice functional approach to aggregating preferences when the aggregating mechanism is required to satisfy the principles of equity and liberty, is considered in the next section.

6. LIBERTY, EQUITY, AND THE POSSIBILITY OF CONSISTENT CHOICE.

The following proposition is true.

Theorem 6.1. Let $f:(\tilde{\Sigma}_o)^{\#N} \rightarrow \Sigma_o$ be a GSAR which satisfies conditions WML and MWE (we know, from Theorem 4.5, that such an f exists), and let C be a CCCF which is H-generated by f. Suppose X contains a subset $A=\{x,y,w,z\}$ such that $(x,y)\in D_j$ and $(w,z)\in D_k$ for some $j,k,\in N$, and either

(i) $\#(\{x,y\}\cap\{w,z\})=1$

or

(ii) $\{x,y\}\cap\{w,z\} = (\{x,y\}\times\{w,z\})\cap D_j = (\{x,y\}\times\{w,z\})\cap D_k = \emptyset$ and $(\{x,y\}\times\{w,z\})\cap D_{j,k} \neq \emptyset$.

Then, there exists a profile $(\bar{P}_i)_{i\in N}\in(\tilde{\Sigma}_o)^{\#N}$ such that an equity-dominated or a liberty-dominated social state can be expelled from the choice set of A only at the cost of rendering the choice set null.

Proof. See Appendix.

Remark 6.2. The impossibility result of Theorem 4.4 can be avoided by weakening the liberty and equity axioms (Theorem 4.5). While preferences can be vague, it seems reasonable to require that choice be exact; and Theorem 6.1 shows that the price to be paid for the possibility result embodied in Theorem 4.5 is that the equity and liberty principles could altogether lose their cutting edge: if soical choice is not to be vacuous, then for certain preference profiles the choice set woud have to admit alternatives that are equity-and liberty-dominated. (In this connection, see Sen's (1976) remarks on the consequences of weakening the liberty principle along the lines suggested by Karni (1978)).

7. CONCLUDING OBSERVATIONS

In this paper, I have examined the problem of aggregating individual preferences when the preference aggregation mechanism is required to satisfy the ethical principles of individual liberty and equity. Subramanian (1987) offers a fuzzy generalization of Sen's (1970) result on 'Paretarian liberalism'; the present paper offers a vague generalization of a result on 'egalitarian liberalism'. The problem has been investigated in both 'relation-functional' and 'choice-functional' terms. The results of these exercises have been essentially negative. This may or may not possess 'surprise-value', but it is worth emphasizing that the results are discouraging despite the fact that:

a) the conventional tight framework of exact or crisp social preferences has been relaxed by allowing for ordinal soft social preferences: this easing-up does not prevent a generalization of the impossibility result from the crisp framework to its soft counterpart;

b) the equity principle explicitly allows for interpersonal comparability of utility: the Arrow-Sen programme of 'extended sympathy', designed to enrich the informational basis of preference aggregation, is not of great assistance in preventing the libertarian dilemma discussed in this paper (in this connection, see also Kelly, 1976); and

c) Hammond's equity principle has been reformulated in such a way that the modified axiom empoyed in this paper is no longer a 'welfarist' principle: from an interpretational point of view, the blame for the 'imposibility of an egalitarian liberal' cannot - unlike in the case of Sen's (1970) 'impossibility of a Paretain liberal' - be laid at the door of 'welfarism'.

In pursuance of the last point, it may be noted that Pattanaik (1988) makes a similar point to the effect that libertarian dilemmas are not necessarily always a reflection of the inadequacies of welfarism. This he does by pointing to the possibility of a problem of internal consistency of libertarian values, interpreted as a conflict between 'group'and 'individual' liberties (in this connection, see also Riley, 1990). Indeed, the 'impossibility of egalitarian liberalism' considered in this paper could itself be construed as an instance of internal inconsistency of libertarian values of the type exemplified by incompatibility between 'group and 'individual' rights - if the 'rules of association' of the group constituted by any two individuals required that the group's collective preference will be determined by considerations of equity (rather than, as is more commonly presumed, by considerations of unanimity).

Finally, it is important to underline the fact that the number of indivudals in society has, throughout this paper, been assumed to be at least two. In a substantive sense, the significance of the 'impossibility of an egalitarin liberal' would greatly diminish if we were to restrict the number of individuals in society to be strictly greater than two; for, with $\#N \geq 3$ it can be verified that the equity principle (in its strong form) would run into problems of internal consistency - which would tend to reduce the force of an inconsistency between the equity and the liberty principles. But in a framework which does allow for a two-person society, there woud appear to be a genuine 'equity-liberty' dilemma.

APPENDIX

PROOFS OF RESULTS IN THE TEXT

Theorem 4.4. There exists no GSAR $f:(\tilde{\Sigma}_{oc})^{\#N} \rightarrow \Sigma_o$ satisfying conditions ML and ME if X contains a subset $A=\{x,y,w,z\}$ such that $(x,y)\in D_j$ and $(w,z)\in D_k$ for some $j,k\in N$, and either

(i) $\#(\{x,y\}\cap\{w,z\})=1$

or

(ii) $\{x,y\}\cap\{w,z\} = (\{x,y\}\times\{w,z\})\cap D_j = (\{x,y\}\times\{w,z\})\cap D_k = \emptyset$ and $(\{x,y\}\times\{w,z\})\cap D_{j,k} \neq \emptyset$.

Proof. Two cases must be distinguished: (i) the doubletones $\{x,y\}$ and $\{w,z\}$ have exactly one alternative in common (say, y=w); and (ii) each of the alternatives x,y,w and z is distinct, $(\{x,y\}\times\{w,z\})\cap D_j = (\{x,y\}\times\{w,z\})\cap D_k = \emptyset$, and $(\{x,y\}\times\{w,z\})\cap D_{j,k} \neq \emptyset$. In what follows, I shall invoke individual preferences which, it can be verified, are compatible with the domain restriction specified in the statement of Theorem 4.4.

Case (i). Consider the following pattern of individual preferences:

$\tilde{P}_j((x,j),(y,j))=d_m$, $\tilde{P}_j((y,j),(x,j))=d_1$; $\tilde{P}_k((y,k),(z,k))=d_m$,

$\tilde{P}_k((z,k),(y,k))=d_1$; $\tilde{P}_j((x,j),(z,j))=d_1$, $\tilde{P}_j((z,j),(x,j))=d_m$;

$\tilde{P}_k((x,k),(z,k))=d_m$, $\tilde{P}_k((z,k),(x,k))=d_1$; $\forall i\in N:\tilde{P}_i((x,k),(x,j))$

$=\tilde{P}_i((z,k),(z,j))=d_m$ & $\tilde{P}_i((x,j),(x,k))= \tilde{P}_i((z,j),(z,k))=d_1$.

By condition ML for individual j over the pair (x,y):

(4.4.1) $P(x,y)\bar{Q}P(y,x)=d_1$.

By condition ML for individual k over the pair (y,z):

(4.4.2) $P(y,z)\bar{Q}P(z,y)=d_1$.

By max-min transitivity over the triple $\{x,y,z\}$:

$P(x,z)Qm(P(x,y),P(y,z))$ whence, in view of (4.4.1) and (4.4.2),

(4.4.3) $P(x,z)\bar{Q}d_1$.

However, since $(x,y)\in D_j$ and $(y,z)\in D_k$, it follows that $(x,z)\in D_{j,k}$, so that by condition ME over the pair (z,x), one has:

(4.4.4) $P(z,x)\bar{Q}\ P(x,z)=d_1$.

(4.4.3) and (4.4.4) are mutually incompatible.

Case (ii). Let individual preferences be as follows:

$\tilde{P}_j((x,j),(y,j))=d_m$, $\tilde{P}_j((y,j),(x,j))=d_1$; $\tilde{P}_k((w,k),(z,k))=d_m$,

$\tilde{P}_k((z,k),(w,k))=d_1$; $\tilde{P}_j((y,j),(w,j))=d_m$, $\tilde{P}_j((w,j),(y,j))=d_1$;

$\tilde{P}_k((y,k),(w,k))=d_1$, $\tilde{P}_k((w,k),(y,k))=d_m$; $\forall i\in N$:$\tilde{P}_i((y,k),(y,j))$

$=\tilde{P}_i((w,k),(w,j))=d_m$ & $\tilde{P}_i((y,j),(y,k))=\tilde{P}_i((w,j),(w,k))=d_1$;

$\tilde{P}_j((x,j),(z,j))=d_m$, $\tilde{P}_j((z,j),(x,j))=d_1$; $\tilde{P}_k((x,k),(z,k))=d_1$,

$\tilde{P}_k((z,k),(x,k))=d_m$; $\forall i\in N$: $\tilde{P}_i((x,j),(x,k))=\tilde{P}_i((z,j),(z,k))=d_m$

& $\tilde{P}_i((x,k),(x,j))=\tilde{P}_i((z,k),(z,j))=d_1$.

It should first be noted that when x,y,w and z are all distinct, the condition $[(\{x,y\}\times\{w,z\})\cap D_j = (\{x,y\}\times\{w,z\})\cap D_k = \emptyset$ & $(\{x,y\}\times\{w,z\})\cap D_{j,k} \neq \emptyset]$ guarantees that $(x,z)\in D_{j,k}$ and $(y,w)\in D_{j,k}$. (This is rather easily verified.) Now, by condition ML to person j over the pair (x,y):

(4.4.5) $P(x,y)\bar{Q}P(y,x)=d_1$.

By condition ME over the pair (y,w):

(4.4.6) $P(y,w)\bar{Q}P(w,y)=d_1$.

By max-min transitivity over the triple $\{x,y,w\}$:

$P(x,w)Q\ m(P(x,y),P(y,w))$ whence, in view of (4.4.5) and (4.4.6), we have:

(4.4.7) $P(x,w)\bar{Q}\ d_1$.

By condition ML to person k over the pair $\{w,z\}$:

(4.4.8) $P(w,z)\bar{Q}\ P(z,w)=d_1$.

By max-min transitivity over the triple $\{x,w,z\}$:

$P(x,z)Q\ m((P(x,w),P(w,z))$ whence, in view of (4.4.7) and (4.4.8), we have:

(4.4.9) $P(x,z)\bar{Q}\ d_1$.

By condition ME over the pair (z,x):

(4.4.10) $P(z,x)\bar{Q}\ P(x,z)=d_1$.

(4.4.9) and (4.4.10) are mutually incompatible, and this completes the proof of the theorem. (Q.E.D).

Theorem 4.5. There exists a GSAR $f:(\bar{\Sigma}_o)^{\#N} \to \Sigma_o$ satisfying conditions WML and MWE.

Proof. Construct the following GSAR f^*:

$\forall x,y\in X, \forall(\tilde{P}_i)_{i\in N} \in(\bar{\Sigma}_o)^{\#N}: P^*(x,y)=d_1$.

It is easy to see that f^* satisfies condition MWE. Let (j,k) be any pair of individuals and (x,y) any pair of alternatives such that $\tilde{P}_k((x,j),(y,j))=d_m$, $\tilde{P}_j((y,j),(x,j))=d_1$, $\tilde{P}_k((x,k),(y,k))=d_1$, $\tilde{P}_k((y,k),(x,k))=d_m$ and $\forall i\in N:\tilde{P}_i((x,k),(x,j))=\tilde{P}_i((y,k),(y,j))=d_m$ & $\tilde{P}_i((x,j),(x,k))=\tilde{P}_i((y,j),(y,k))=d_1$. Then, by construction of f^*, $P^*(x,y)$ $(=d_1)Q$ $P^*(y,x)(=d_1)$, as required by condition MWE. It is also clear that f^* satisfies condition WML. For, again, let j be any individual and (x,y) any pair of alternatives such that $\tilde{P}_j((x,j),(y,j))=d_m$ and $\tilde{P}_j((y,j),(x,j))=d_1$; then, by construction of f^*, $P^*(x,y)$ $(= d_1)QP^*(y,x)$ $(= d_1)$, as required for condition WML to be satisfied. It is immediate that P^* is irreflexive, and also satisfies asymmetry by default (note that by construction of f^*, $P^*(x,y)=d_m$ can happen for no $x,y\in X$.) Finally, consider any triple of alternatives $\{x,y,z\}$. By construction of f^*, $P^*(x,z)(=d_1)Q\ m[P^*(x,y)(=d_1),\ P^*(y,z)(=d_1)]$: thus, P^* satisfies max-min transitivity. This completes the proof of the theorem. (Q.E.D).

Lemma 5.5. Let C be a CCF. If C is H-rationalizable by a SBPR $P\in\Sigma_o$, then C satisfies property β^+.

Proof. I shall suppose the lemma to be false and derive a contradiction. Let C be a CCF which is H-rationalizable be a $P\in\Sigma_o$. Let $x,y\in X$ and $A,B\in\mathcal{X}$ with $A \subseteq B$ such that $x\in C(A)$, $y\in A$, $y\in C(B)$ and $x\notin C(B)$: this will be shown to lead to contradiction. Since $x\in C(A)$ and $y\in A$, and C is H-rationalizable by P,

(5.5.1) $P(x,x_A^*)QP(y,y_A^*)$.

Simililarly, since $y \in C(B)$, $x \in B$, $x \notin C(B)$, and C is H-rationalizable by P,

(5.5.2) $P(y,y^*_B)\bar{Q}\ P(x,x^*_B)$.

Note next that since $A \subseteq B$,

(5.5.3) $P(y,y^*_A)Q\ P(y,y^*_B)$.

From (5.5.1),(5.5.2) and (5.5.3), we obtain

(5.5.4) $P(x,x^*_A)\bar{Q}\ P(x,x^*_B)$.

Since $P \in \Sigma_o$, P satisfies max-min transitivity; consequently, max-min transitivity over the triple $\{x,y,x^*_B\}$ requires that

(5.5.5) $P(x,x^*_B)Q\ m(P(x,y),P(y,x^*_B))$.

Suppose now that

(5.5.6) $P(y,x^*_B)Q\ P(x,y)$.

Then, by (5.5.5), we must have

(5.5.7) $P(x,x^*_B)Q\ P(x,y)$.

But note that, by definition of x^*_A,

(5.5.8) $P(x,y)Q\ P(x,x^*_A)$.

From (5.5.7) and (5.5.8), we obtain: $P(x,x^*_B)QP(x,x^*_A)$ which, however, contradicts (5.5.4). Therefore, (5.5.6) must be false, and one must have:

(5.5.9) $P(x,y)\bar{Q}\ P(y,x^*_B)$ which, in view of (5.5.5), leads to:

(5.5.10) $P(x,x^*_B)Q\ P(y,x^*_B)$.

But note that, by definition of y^*_B,

(5.5.11) $P(y,x^*_B)Q\ P(y,y^*_B)$.

From (5.5.10) and (5.5.11), we obtain:

$P(x,x^*_B)Q\ P(y,y^*_B)$ which, however, contradicts (5.5.2). (Q.E.D)

Theorem 6.1. Let $f:(\tilde{\Sigma}_o)^{\#N} \to \Sigma_o$ be a GSAR which satisfies conditions WML and MWE (we know, from Theorem 4.5, that such an f exists), and let C be a CCCF which is H-generated by f. Suppose X contains a subset $A=\{x,y,w,z\}$ such that $(x,y) \in D_j$ and $(w,z) \in D_k$, for some $j,k \in N$, and either

(i) $\#(\{x,y\} \cap \{w,z\}) = 1$

or

(ii) $\{x,y\} \cap \{w,z\} = (\{x,y\} \times \{w,z\}) \cap D_j = (\{x,y\} \times \{w,z\}) \cap D_k = \emptyset$ and $(\{x,y\} \times \{w,z\}) \cap D_{j,k} \neq \emptyset$.

Then, there exists a profile $(\tilde{P}_i)_{i \in N} \in (\tilde{\Sigma}_o)^{\#N}$ such that an equity-dominated or a liberty-dominated social state can be expelled from the choice set of A only at the cost of rendering the choice set null.

Proof. Two cases must be distinguished: (i) the pairs (x,y) and (w,z) have exactly only alternative in common, say y=w; and (ii) $\{x,y\}\cap\{w,z\} = (\{x,y\}\times\{w,z\})\cap D_j = (\{x,y\}\times\{w,z\})\cap D_k = \emptyset$ and $(\{x,y\}\times\{w,z\})\cap D_{j,k} \neq \emptyset$.

Case (i). Consider exactly the same pattern of individual preferences as has been employed in the proof of case (i) of Theorem 4.4. It is easy to see, given the pattern of individual preferences, that x is an equity-dominated state (dominated by z), while y and z are liberty-dominated states (dominated by x and y respectively). It must now be proved that: if $x\notin C(A)$, then $C(A) = \emptyset$; if $y\notin C(A)$, then $C(A) = \emptyset$; and if $z\notin C(A)$, then $C(A) = \emptyset$.

Suppose, to the contrary, that - first - $x\notin C(A)$ and $C(A) \neq \emptyset$. Then at least one of the following must be true: (ia) $y\in C(A)$ or (ib) $z\in C(A)$. By WML to person j over the pair (x,y), $P(x,y)Q\ P(y,x)$ whence, since C is H-generated by f, $x\in C(\{x,y\})$. Recalling that the range of f is Σ_o, we know from Lemma 5.5. that C satisfies Property β^+. Therefore, $[x\in C(\{x,y\})$ and $y\in C(A)]$ implies $x\in C(A)$, contrary to supposition. Hence (ia) cannot be true. Suppose, next, (ib) to be true. By WML to person k over the pair (y,z), $P(y,z)QP(z,y)$, whence $y\in C(\{y,z\})$. By property β^+, $[y\in C(\{y,z\})$ and $z\in C(A)]$ implies $y\in C(A)$ which, as we have already seen, leads to contradiction. Hence, (ib) is also false, and $C(A)=\emptyset$, as desired.

Next suppose $y\notin C(A)$ and $C(A)\neq\emptyset$. Then, at least one of the following is true: (iia) $x\in C(A)$ or (iib) $z\in C(A)$. Suppose, first, (iia) to be true. By condition MWE over the pair of alternatives (z,x), $P(z,x)Q\ P(x,z)$, whence $z\in C(\{x,z\})$. Since, again by Lemma 5.5, C satisfies property β^+, $[z\in C(\{x,z\})$ and $x\in C(A)]$ implies $z\in C(A)$. Note, further, that by WML to k over the pair (y,z), $P(y,z)Q\ P(z,y)$, so $y\in C(\{y,z\})$. Again by Property β^+, $[y\in C(\{y,z\})$ and $z\in C(A)]$ implies $y\in C(A)$, contrary to supposition. Therefore, (iia) is false. Suppose (iib) to be true. But we have already seen that $z\in C(A)$ must imply, contrary to supposition, that $y\in C(A)$. Therefore, (iib) is also false, that is, $C(A)=\emptyset$, as desired.

Finally, suppose $z \notin C(A)$ and $C(A) \neq \emptyset$. Then, at least one of the following must be true. (iiia) $x \in C(A)$ or (iiib) $y \in C(A)$. Suppose (iiia) to be true. By MWE over (z,x), $P(z,x)QP(x,z)$; hence $z \in C(\{x,z\})$. By property β^{+}, $[z \in C(\{x,z\})$ and $x \in C(A)]$ implies $z \in C(A)$ which is a contradiction. Thus, (iiia) must be false, leaving us with (iiib). By WML to j over (x,y), $P(x,y)QP(y,x)$, leading to: $x \in C(\{x,y\})$ and, by property β^{+}, to $x \in C(A)$; but we have just seen that $x \in C(A)$ cannot be true. Therefore, (iiib) is also false, leaving us with $C(A)=\emptyset$, as desired.

Case (ii). By invoking exactly the same configuration of individual preferences as has been done in the proof of case (ii) of Theorem 4.4, case(ii) of the present theorem can be easily proved along the lines of the proof of case (i); the proof is here omitted in order to avoid tedious repetitiveness. (Q.E.D.).

REFERENCES

Arrow,K.J. (1959): 'Rational Choice Functions and Orderings', Economica, 26, 121-127.

Arrow,K.J. (1963): Social Choice and Individual Values (second edition). New York: John Wiley and Sons.

Arrow,K.J. (1977): 'Extended Sympathy and the Possibility of Social Choice', American Economic Review, 67.

Barrett,C.R. and P.K.Pattanaik (1990): 'Aggregation of Fuzzy Preferences', in J.Kacprzyk and M.Fedrizzi (eds.): Multiperson Decision Making using Fuzzy Sets and Possibility Theory. Kluwer Academic Publishers: printed in The Netherlands.

Barrett,C.R., P.K.Pattanaik and M.Salles (1990): 'On the Structure of Fuzzy Social Welfare Functions', Fuzzy Sets and Systems, 19, 1-10.

Barrett,C.R., P.K.Pattanaik and M.Salles (1990): 'On Choosing Rationally When Preferences Are Fuzzy', Fuzzy Sets and Systems, 34, 197-212.

Basu,K. (1984): 'Fuzzy Revealed Preference Theory', Journal of Economic Theory, 32, 212-227.

Basu,K., R.Deb and P.K.Pattanaik (1988): 'Soft Sets: An Ordinal Formulation of Vagueness with Some Applications to the Theory of Choice', forthcoming in Fuzzy Sets and Systems.

Bordes,G. (1970): 'Consistency, Rationality and Collective Choice', Review of Economics Studies, 43, 444-457.

Dutta,B. (1987): 'Fuzzy Preferences and Social Choice', Mathematical Social Sciences.

Dutta,B., S.C.Panda and P.K.Pattanaik: 'Exact Choice and Fuzzy Preferences', Mathematical Social Sciences, 11, 53-68.

Gibbard,A. (1974): 'A Pareto Consistent Libertarian Claim', Journal of Economic Theory, 7, 399-410.

Goguen,J.A. (1967): 'L-fuzzy Sets', Journal of Mathematical Analysis and Applications, 18.

Hammond,P.J. (1976): 'Equity, Arrow's Conditions, and Rawls' Difference Principle', Econometrica, 44.

Karni,E. (1978): 'Collective Rationality, Unanimity and Liberal Ethics', Review of Economic Studies, 45, 571-574.

Kelly,J.S. (1976): 'The Impossibility of a Just Liberal', Economica.

Pattanaik, P.K. (1988): 'On the Consistency of Libertarian Values', Economica.

Richter, M. (1966): 'Revealed Preference Theory' Econometrica, 34, 635-645.

Sen, A.K. (1970): 'The Impossibility of a Paretian Liberal', Journal of Political Economy, 78, 152-159.

Sen, A.K. (1971): 'Choice Functions and Revealed Preference', Review of Economic Studies, 38, 307-317.

Sen, A.K. (1973): On Economic Inequality. Oxford: Clarendon.

Sen, A.K. (1976): 'Liberty, Unanimity and Rights', Economica, 43, 217-245.

Sen, A.K. (1977): 'Social Choice Theory: A Re-examination', Econometrica, 45, 53-89.

Sen, A.K. (1979): 'Personal Utilities and Public Judgments or What's Wrong with Welfare Economics', Economic Journal, 89, 537-558.

Subramanian, S. (1987): 'The Liberal Paradox with Fuzzy Preferences', Social Choice and Welfare, 4.