Title of Paper: Aggregation of Fuzzy Preferences: Variations on a Theme.

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Abstract of Paper:

In this paper, I explore alternative constructions of the asymmetric and symmetric components of a fuzzy weak binary preference relation. For these alternative constructions, I examine the 'performance' of Arrow's General Possibility Theorem in a framework of vague individual and social preferences.

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Aggregation of Fuzzy Preferences: Variations on a Theme

by

S. Subramanian

1. INTRODUCTION

Barrett, Pattanaik and Salles (1986) have considered the classical Arrow (1963) problem of aggregation of individual preferences, in a context wherein both personal and public preferences are taken to be vague. The results of these exercises have been, on the whole, discouraging, with the impossibilities in the exact framework more or less carrying over to their fuzzy counterparts. In the works just cited, an important component of the methodology employed has been the treatment of the fuzzy binary relation of strict preference as a primitive. Dutta (1987) has undertaken the task of starting out with the fuzzy relation of 'weak' preference and then deriving, through axiomatic rationalization, the asymmetric and symmetric parts of this relation. He then proceeds, within this framework of fuzzy individual and social preferences, to review Arrow-type theorems of aggregation. He discovers that his results are more encouraging than the Barret-Pattanaik-Salles conclusions.

In the present paper I too start out with a fuzzy weak preference relation and, with a slightly differently specified axiom system to that employed by Dutta, derive the corresponding strict preference and indifference relations. Within this framework of fuzzy preferences, the 'relation-functional' versions of some Arrow-type aggregation problems ar re-examined. As we shall see, the results of these exercises are, at best, 'mixed'; and, at worst, essentially negative.

2. THE ASYMMETRIC AND SYMMETRIC COMPONENTS OF THE WEAK FUZZY PREFERENCE RELATION: ALTERNATIVE CONSTRUCTIONS

$X = \{x,y,z,\ldots\}$ is the finite set of all conceivable <u>alternatives</u>, with X containing at least three elements.

$N = \{1,\ldots,i,\ldots,n\}$ is the finite set of <u>individuals</u> constituting society, with N containing at least two members.

A <u>fuzzy weak binary preference relation</u> (FWBPR) is a function $R: X\times X \rightarrow [0,1]$, while an <u>exact weak binary preference relation</u> (EWBPR) is a function $R: X\times X \rightarrow \{0,1\}$. An FWBPR R on X is (a) <u>reflexive</u> iff $\forall\ x\in X$: $R(x,x)=1$; (b) <u>connected</u> iff for all distinct $x,y,\in X$: $R(x,y)+R(y,x)\geq 1$; and (c) <u>maxmin</u> (or <u>M</u>-) <u>transitive</u> iff for all distinct $x,y,z\in X$: $R(x,z)\geq \min[R(x,y),R(y,z)]$. A fuzzy weak binary preference <u>ordering</u> (FWBPO) is an FWBPR which is reflexive, connected and M-transitive.

Dutta (1987) has shown that extracting the strict preference (P) and indifference (I) relations from an FWBPR R on X is an unfruitful exercise when R,P and I are governed by the rules which hold for them in the exact framework. The following result is due to him:

<u>Theorem 1.</u> Let R be a connected FWBPR on X satisfying

(i) $R = P\cup I$, viz. $\forall x,y,\in X$: $R(x,y)=\max[P(x,y),I(x,y)]$;

(ii) I is symmetric, viz. $\forall x,y\in X$: $I(x,y)=I(y,x)$;

(iii) P is (strongly) asymmetric, viz. $\forall x,y\in X$: $P(x,y) > 0 \rightarrow P(y,x)=0$; and

(iv) $P\cap I = \emptyset$, viz. $\forall x,y\in X$: $\min[P(x,y),I(x,y)]=0$.

Then, either R is an EWBPR, or, $\forall x,y\in X$: $R(x,y)=R(y,x) = I(x,y)=I(y,x)$.

Proof. Dutta (1987; Proposition 2.4). (Q.E.D)

Clearly, in order to find some meaningful way out of Theorem 1, one or more of axioms (i) - (iv) listed in the statement of the theorem must be relaxed. Dutta himself regards axiom (iv) as a natural candidate for relaxation, and states and proves the following result:

Theorem 2. Let R be a connected FWBPR satisfying

(i) $R = P \cup I$;

(ii) I is symmetric;

(iii) P is (strongly) asymmetric; and

(iv′) $\forall x,y \in X : R(x,y) = R(y,x) \rightarrow P(x,y) = P(y,x)$.

Then, $\forall x,y \in X$:

$$\left.\begin{array}{l} P(x,y) = R(x,y) \text{ if } R(x,y) > R(y,x); \\ \quad = 0, \text{ otherwise; and} \\ I(x,y) = \min\,[R(x,y), R(y,x)]. \end{array}\right\} \quad \ldots(1)$$

Proof. Dutta (1987; Proposition 2.5). (Q.E.D.).

In what follows, I pursue an alternative route to weakening the axiom system (i) - (iv). In particular, it is easy to see that axiom (iii) is just about the strongest version of asymmetry one can invoke - requiring, as it does, that no two alternatives can be strictly preferred to each other with a positive degree of confidence. Barrett, Pattanaik and Salles (1986) employ a weaker condition of asymmetry in terms of which if any alternative x is preferred to any other alternative y with complete confidence, then y may not be preferred to x with any positive degree of confidence: $\forall x,y \in X: P(x,y)=1 \rightarrow P(y,x)=0$. Under this formulation, given a pair of alternatives x,y, one can have : $P(x,y) = P(y,x) = .99$; if now the extent to which x is strictly preferred to y were to rise by a marginal degree from .99 to 1.0, then asymmetry would demand that P(y,x) must decline abruptly from .99 to zero. This sort of discontinuously precipitous decline in P(y,x) for a marginal increase in $P(x,y)$ is not a very appealing property of the asymmetric relation. Just about the weakest asymmetry condition one can invoke, and which escapes the difficulty just discussed, is one in which any pair of distinct alternatives can be preferred to each other with a positive degree of confidence, provided only that each alternative is not preferred to the other with complete confidence: $\forall x,y \in X: P(x,y) = 1 \rightarrow P(y,x) < 1$. If we relax only axiom (iii) in the axiom system (i) - (iv), along the lines just discussed, we obtain the following result:

Theorem 3. Let R be a connected FWBPR satisfying

(i) $R = P \cup I$;

(ii) I is symmetric;

(iii′) P is (weakly) asymmetric, viz. $\forall x,y \in X: P(x,y) = 1 \rightarrow P(y,x) < 1$; and

(iv) $P \cap I = \emptyset$.

Then, either 2(A) or 2(B) below is true:

$$\left.\begin{array}{l} \forall x,y \in X: P(x,y) = 0 \;\&\; I(x,y) = R(x,y) \text{ if } R(x,y) = R(y,x) = 1; \\ \quad \text{and} \\ \quad P(x,y) = R(x,y) \;\&\; I(x,y) = 0, \text{ otherwise.} \end{array}\right\} \quad 2(a)$$

$$\left.\begin{array}{l} \forall x,y \in X: P(x,y) = 0 \;\&\; I(x,y) = R(x,y) \text{ if } R(x,y) = R(y,x); \\ \quad \text{and} \\ \quad P(x,y) = R(x,y) \;\&\; I(x,y) = 0, \text{ otherwise.} \end{array}\right\} \quad 2(b)$$

Proof Suppose, first, that $R(x,y) = R(y,x) = 1$ but $P(x,y) \neq 0$. Then, by (iv), $I(x,y)=0$; by (ii), $I(y,x)=0$; and by (i), $R(x,y)=P(x,y)$ and $R(y,x)=P(y,x)$. We therefore have $P(x,y) = P(y,x) = 1$ which however violates (iii′). Therefore, if $R(x,y) = R(y,x) = 1$, then $P(x,y) = P(y,x) = 0$ and, by virtue of (i), $I(x,y) = R(x,y)$, as required. Suppose, next, that $\sim[R(x,y) = R(y,x) = 1]$. We distinguish two cases: (a) $R(x,y) \neq R(y,x)$; and (b) $R(x,y) = R(y,x) < 1$. Suppose (a) is true but $P(x,y) \neq R(x,y)$. Then, by

(i) and (iv), $R(x,y) = I(x,y) > 0$; by (ii), $I(x,y) = I(y,x)$; and by (i) and (iv) again, $R(y,x) = I(y,x)$. So we have $R(x,y) = R(y,x)$ which contradicts (a). Suppose (b) is true. We have to show that either $P(x,y) = R(x,y)$ (and therefore, by (iv), $I(x,y)=0$) or $P(x,y)=0$ (and therefore, by (i), $I(x,y)=R(x,y)$). Suppose, to the contrary, that both (b1) $[P(x,y) \neq R(x,y)]$ and (b2) $[P(x,y) \neq 0]$. But notice that if (b1) is true then by (i), $R(x,y)=I(x,y)$ and by (iv), $P(x,y)=0$, viz. (b2) is false; conversely, if (b2) is true, then by (iv), $I(x,y)=0$ and by (i), $R(x,y)=P(x,y)$, viz. (b1) is false: therefore, under either of (b1) or (b2), R cannot simultaneously satisfy (i) and (iv). This completes the proof of the Theorem. (Q.E.D.).

In what follows we shall, in the context of preference aggregation problems, explore the relative 'success' records of formulations (1), (2A) and (2B) of the asymmetric and symmetric parts of the weak preference relation.

3. FUZZY AGGREGATION RULES

Let T be the set of all FWBPRs on X. In this paper we shall be specifically concerned with three distinguished subsets of T: H_o, H_1 and H_2, of which H_o is the set of all FWBPOs R on X such that the asymmetric and symmetric components of R are as defined in (1); and H_1 (respectively, H_2) is the set of all FWBPOs R on X such that the asymmetric and symmetric parts of R are as defined in 2A (respectively, 2(B)).

A <u>fuzzy aggregation rule</u> (FAR) is a function $f:\hat{T}^n \rightarrow \tilde{T}$ $(\hat{T},\tilde{T}(\neq\emptyset) \subseteq T)$ such that, for every n-tuple of individual FWBPRs (R_i) in its domain, f specifies a unique social FWBPR R in its range.

Elements of $\hat{T}^n$, which are preference profiles, will be designated $(R_i)_{i\in N}$, $(R'_i)_{i\in N}$, etc; and we shall also write R for $f(<R_i>_{i\in N})$, R' for $f(<R'_i>_{i\in N})$, etc.

Some restrictions one may wish to impose on an FAR are defined below (these are fairly standard conditions in the social choice literature and will therefore not be elaborately explained).

An FAR $f:\hat{T}^n \to \tilde{T}$ satisfies

(a) neutrality (condition N) iff $\forall (R_i)_{i\in N}, (R'_i)_{i\in N} \in \hat{T}^n$ and for all distinct $x,y,w,z \in X$:

$[R_i(x,y)=R'_i(z,w)\ \forall i\in N\ \&\ R_i(y,x)=R'_i(w,z)\ \forall i\in N]$ implies
$[R(x,y)=R'(z,w)\ \&\ R(y,x)=R'(w,z)]$;

(b) Independence of Irrelevant Alternatives (Condition I) iff $\forall (R_i)_{i\in N}, (R'_i)_{i\in N} \in \hat{T}^n$ and for all distinct $x,y \in X$:

$[R_i(x,y)=R'_i(x,y)\ \forall i\in N\ \&\ R_i(y,x)=R'_i(y,x)\ \forall i\in N]$ implies

$[R(x,y)=R'(x,y)\ \&\ R(y,x)=R'(y,x)]$;

(c) anonymity (Condition A) iff $\forall (R_i)_{i\in N}, (R'_i)_{i\in N} \in \hat{T}^n$ and for all distinct $x,y \in X$:

$[\exists\ \sigma \in \Sigma$ such that $R_i(x,y)=R'_{\sigma(i)}(x,y)\ \forall i\in N\ \&\ R_i(y,x) = R'_{\sigma(i)}(y,x)\ \forall i\in N]$ implies $[R(x,y)=R'(x,y)\ \&\ R(y,x)=R'(y,x)]$ where Σ is the set of all one-to-one correspondences from N to itself;

(d) non-dictatorship (Condition D) iff there does not exist $j\in N$ such that $\forall (R_i)_{i\in N} \in \hat{T}^n$ and for all distinct $x,y \in X$:

$P_j(x,y) > P_j(y,x) \to P(x,y) > P(y,x)$;

(e) non-oligarchy (Condition O) iff there does not exist a coalition C such that $\forall (R_i)_{i\in N} \in \hat{T}^n$ and for all distinct $x,y \in X$: (i) $P_C(x,y) > P_C(y,x) \to P(x,y) > P(y,x)$; and (ii) $[\exists j\in C: P_j(x,y) > P_j(y,x)] \to P(y,x) = 0$,

where a coalition is any nonempty subset of N and, for all $C(\neq\emptyset)\subseteq N$ and all distinct $x,y\in X$, $[P_C(x,y) > P_C(y,x)]$ is employed as a shorthand for $[P_i(x,y) > P_i(y,x)\ \forall i\in C]$; and

(f) the Pareto criterion (Condition P) iff $\forall (R_i)_{i\in N} \in \hat{T}^n$ and for all distinct $x,y\in X$: $P(x,y) \geq \min_{i\in N} P_i(x,y)$.

It should be noted that an FAR which satisfies neutrality also satisfies independence of irrelevant alternatives, while an FAR which satisfies anonymity also satisfies non-dictatorship. A couple of additional definitions of relevance are provided below.

Almost Decisiveness. A coalition C (resp., individual j) is almost decisive for any x against any y iff

$[P_C(x,y)>P_C(y,x)$ & $P_{N-C}(y,x)>P_{N-C}(x,y)]$ implies $[P(x,y)>P(y,x)]$

{resp., $[P_j(x,y)>P_j(y,x)$ & $P_{N-\{j\}}(y,x)>P_{N-\{j\}}(x,y)]$ implies $[P(x,y)>P(y,x)]$}.

Decisiveness. A coalition C (resp., individual j) is decisive for any x against any y iff

$[P_C(x,y)>P_C(y,x)]$ implies $[P(x,y)>P(y,x)]$

{resp., $[P_j(x,y)>P_j(y,x)]$ implies $[P(x,y)>P(y,x)]$}.

Given the preceding inventory of concepts and definitions, we can proceed to a consideration of some substantive results in preference aggregation.

4. SOME 'ARROW-TYPE' THEOREMS

Dutta (1987; Proposition 3.9) has proved that there exists an FAR $f:H^n_o \rightarrow H_o$ satisfying the Arrow conditions I, D and P. However, an FAR $f:H^n_o \rightarrow H_o$ falls foul of the Gibbardian 'oligarchy' result: Dutta (1987: Remark 3.11) asserts that there exists no FAR $f:H^n_o \rightarrow H_o$ satisfying conditions I,O and P.

Turning next to weak fuzzy preference relations whose asymmetric and symmetric components are as defined in (2A), we begin with a strong existence result:

<u>Theorem 4.</u> There exists an FAR $f:H^n_1 \rightarrow H_1$ satisfying conditions N,A and P.

<u>Proof.</u> Construct the following FAR $\hat{f}$:

$$\forall x,y \in X,\ \forall (R_i)_{i \in N} \in H^n_1 : \hat{R}(x,y)=1 \text{ if } x=y;$$
$$=[1+\min_{i \in N} R_i(x,y)]/2 \text{ otherwise.}$$

Note first that by construction of $\hat{f}$, $\hat{R}$ is reflexive. Moreover, by construction of $\hat{f}$, for all distinct $x,y \in X: \hat{R}(x,y) \geq 1/2$, which ensures the $\hat{R}$ is connected. To see that $\hat{R}$ is M-transitive, consider the following. Let $\{x,y,z\} \subseteq X$ be any triple of distinct alternatives, and let $\min_{i \in N} R_i(x,z)=R_k(x,z)$ for some $k \in N$. Then, since the R_i are M-transitive, one must have:

(4.1) $R_k(x,z) \geq \min\,[R_k(x,y),\ R_k(y,z)]$.

Suppose (a) $R_k(x,y) \geq R_k(y,z)$. Then, by virtue of (4.1),

$R_k(x,z) \geq R_k(y,z)$ [$\geq \min_{i \in N} R_i(y,z)$]. Further, $R_k(x,z) \geq R_k(y,z)$ implies that $R_k(x,z) \geq \min\,[\min_{i \in N} R_i(x,y),\ \min_{i \in N} R_i(y,z)]$: this is true irrespective of whether $\min_{i\in N} R_i(x,y) \geq R_k(y,z)$ or $\min_{i\in N} R_i(x,y) < R_k(y,z)$. If (a) is not true, it must be the case that (b) $R_k(y,z) > R_k(x,y)$ so that, in view of (4.1), $R_k(x,z) \geq R_k(x,y)$ $[\geq \min_{i\in N} R_i(x,y)]$. Further, $R_k(x,z) \geq R_k(x,y)$ implies that $R_k(x,z) \geq \min [\min_{i\in N} R_i(x,y), \min_{i\in N} R_i(y,z)]$: this is true irrespective of whether $\min_{i\in N} R_i(y,z) \geq R_k(x,y)$ or $\min_{i\in N} R_i(y,z) < R_k(x,z)$. We have thus proved that for every triple of distinct alternatives $\{x,y,z\}$:

$\min_{i\in N} R_i(x,z) \geq \min [\min_{i\in N} R_i(x,y), \min_{i\in N} R_i(y,z)]$, whence $[1 + \min_{i\in N} R_i(x,z)]/2 \geq \min [\{1 + \min_{i\in N} R_i(x,y)\}/2, \{1 + \min_{i\in N} R_i(y,z)\}/2]$ or, equivalently, by construction of $\hat{f}$, $\hat{R}(x,y) \geq \min$

$[\hat{R}(x,y), \hat{R}(y,z)]$, as required by M-transitivity. To see that $\hat{f}$ satisfies PC, note that for any pair of distinct alternatives $x,y\in X$, if $\min_{i\in N} P_i(x,y) > 0$, then, by construction of $\hat{f}$, $\hat{P}(x,y) = \hat{R}(x,y) = [1 + \min_{i\in N} R_i(x,y)]/2 \geq \min_{i\in N} R_i(x,y) = \min_{i\in N} P_i(x,y)$.

The proof of the theorem is completed by noting that $\hat{f}$ obviously satisfies both neutrality and anonymity. (Q.E.D.).

Theorem 4 is a strong possibility result, and implies that the Arrow paradox can be circumvented in the fuzzy framework of aggregation under review: this follows from recalling that Condition N implies Condition I and Condition A implies Condition D.

The next result shows that an aggregation rule from H^n_1 to H_1 can get around the Gibbardian 'oligarchy' problem as well:

<u>Theorem 5:</u> There exists an FAR $f: H^n_1 \rightarrow H_1$ satisfying Conditions I,O and P.

<u>Proof.</u> Consider again the FAR $\hat{f}$ constructed in the proof of Theorem 4. We already know that $\hat{f}$ satisfies I and PC, and that $\hat{R}$

is reflexive, connected and transitive, so it only remains to prove that $\hat{f}$ is non-oligarchic. Suppose, instead, that $\hat{f}$ *is* oligarchic. It must be the case that if an oligarchy C exists, then # C>1 since we know $\hat{f}$ (from Theorem 4) to be non-dictatorial. Now consider $(R_i)_{i \in N} \in H^n{}_1$, $x,y \in X$ and $j,k \in C$ such that $R_j(x,y) = R_k(y,x) = 1$ and $R_j(y,x) = R_k(x,$ $= 0$. Then, since C has been presumed to be an oligarchy, j and k must be vetoers, viz. $\hat{P}(x,y) = \hat{P}(y,z) = 0$; however, by construction of $\hat{f}$, $\hat{P}(x,y) = \hat{R}(x,y) = [1 + \min_{i \in N} R_i(x,y)]/2 = [1 + R_k(x,y)]/2 = \frac{1}{2}$, and we have a contradiction. Therefore, our supposition regarding the existence of an oligarchy is false, and the theorem is proved. (Q.E.D.).

While Theorems 4 and 5 are distinctly encouraging, it is disappointing to note that precisely the converse is true when we employ Construction (2B) of the asymmetric and symmetric parts of the weak fuzzy preference relation; and this despite the fact that (2B) shares with (2A) the common parentage of the system of axioms [(i),(ii),(iii') and (iv)]. This is the content of the following result (the proof of which, by virtue of its length, has been relegated to an Appendix at the end of this paper).

Theorem 6. There exists no FAR $f: H^n{}_2 \to H_2$ satisfying Conditions I, D and P.

Proof. Appendix. (Q.E.D.).

5. CONCLUDING OBSERVATIONS

Is there any *a' priori* criterion for choosing among Constructions (1), (2A) and (2B) of the symmetric and asymmetric parts of the weak preference relation? One *possible* consideration that may have some weight is to require that constructions in the fuzzy framework should mimic the properties of their counterparts in the exact framework as closely as possible - so that vague constructions are seen to be, as nearly as possible, genuine generalizations of the canonical settings. In the context of the relationship between the weak and the strict preference relations R and P, the following two properties - which one may call Properties I and II respectively - hold in the exact framework.

Property I. $\forall x,y \in X$: $R(x,y)=R(y,x) \rightarrow P(x,y)=P(y,x)=0$.

Property II. $\forall x,y \in X$: $P(x,y)-P(y,x)=R(x,y)-R(y,x)$.

Now it is immediately clear that Constructions (1) and (2B) satisfy Property I, which (2A) does not. Again, Constructions (2A) and (2B) satisfy Property II, while (1) does not - as is revealed in Figure 1 which plots the graph of $P(x,y) - P(y,x)$ as a function of $R(x,y) - R(y,x)$ (in the diagram, the value of $R(y,x)$ has been fixed at some $\alpha \in (0,1)$).

If, therefore, Constructions (1) and (2A) are disqualified, we are left with (2B), which is the only one of the three constructions that satisfies both Properties I and II. However, Construction, (2B) is the perfect recipe for 'aggregational impossibility'! There appears, therefore, to be no easy fuzzy way out of the exact nihilism of Arrow's theorem.

APPENDIX

Proof of Theorem 6

Theorem 6. There exists no FAR $f:H^n_2 \to H_2$ satisfying conditions I,D and P.

We shall prove the theorem exactly along the lines employed by Sen (1970) in proving the Arrowian General Possibility Theorem in the exact framework. But first, three lemmata to this end.

Lemma 1. Let R be a reflexive and connected FWBPR on X. If R is M-transitive, then R is M′-transitive, where M′-transitivity requires that for any triple of distinct alternatives $\{x,y,z\} \subseteq X$: $[R(x,y) > R(y,x)$ & $R(y,z) > R(z,y)]$ implies $[R(x,z) > R(z,x)]$.

Proof Suppose the lemma to be false. Then, there exists a triple $\{x,y,z\} \subseteq X$ such that R is M-transitive over the triple but not M′-transitive, viz. $R(x,y) > R(y,x)$ & $R(y,z) > R(z,y)$, and $R(z,x) \geq R(x,z)$.

Since R satisfies M-transitivity,

(L1.1) $R(x,z) \geq \min[R(x,y),R(y,z)]$.
Assume WLOG that

(L1.2) $\min[R(x,y),R(y,z)]=R(y,z)$.
Then, (L1.1) and (L1.2) yield

(L1.3) $R(x,z) \geq R(y,z)$,
whence, since $R(y,z) > R(z,y)$ by data,

(L1.4) $R(x,z) > R(z,y)$.

By M-transitivity over $\{z,x,y\}$, $R(z,y) \geq \min[R(z,x),R(x,y)]$ which, in view of (L1.4) and the hypothesis that $R(z,x) \geq R(x,z)$ implies that

(L1.5) $R(z,y) \geq R(x,y)$.

By M-transitivity over $\{y,z,x\}$, $R(y,x) \geq \min[R(y,z),R(z,x)]$ which, in view of (L1.3) and the hypothesis that $R(z,x) \geq R(x,z)$, leads to

(L1.6) $R(y,x) \geq R(y,z)$.

From (L1.5) and (L1.6), and recalling that $R(y,z) > R(z,y)$ by data, one has: $R(y,x) > R(x,y)$ - which, however, contradicts the data. This completes the proof of the lemma. (Q.E.D.).

Before stating the next result, a notational clarification:
Notation. Suppose $\exists j \in N$ such that j is almost decisive (respectively, decisive) over some ordered pain of distinct alternatives $(x,y) \in X^2$. Then, we shall denote this by the notation $D_j(x,y)$ [respectively, $\bar{D}_j(x,y)$].

Lemma 2. Let $f: H^n{}_2 \to H_2$ be an FAR satisfying conditions P and I. If there exists $j \in N$ such that $D_j(x,y)$ for some $(x,y) \in X^2$, then it is true that $\bar{D}_j(u,v)$ for all ordered pairs of distinct alternatives $(u,v) \in X^2$.

Proof. Suppose $D_j(x,y)$ for some $(x,y) \in X^2$. Let z be any other alternative. It will first be shown that j is decisive over every pair of alternatives in the set $\{x,y,z\} \times \{x,y,z\}$, viz. that j is decisive over the ordered pairs (x,y), (y,x), (y,z), (z,y), (x,z) and (z,x). To this end, consider the following. (In what follows, it is readily verifiable that every profile invoked is indeed a member of the domain $H^n{}_2$ of the FAR f. Moreover, use will be recurrently made of Lemma 1 which assures us that, since every R in the range H_2 of f is M-transitive, it is also M'-transitive).

Suppose, first, that $R_j(x,y)=1, R_j(y,x)=0, R_j(y,z)=1, R_j(z,y)=0, R_j(x,z)=1, R_j(z,x)=0$, $R_{N-\{j\}}(x,y)=0$, $R_{N-\{j\}}(y,x)=1$, $R_{N-\{j\}}(y,z)=1$ and $R_{N-\{j\}}(z,y)=0$. Then, since $D_j(x,y)$, $P(x,y) > P(y,x)$, viz. $R(x,y) > R(y,x)$; and by PC, $P(y,z)=1 > P(z,y)$, viz. $R(y,z)=1 > R(z,y)$. Since R is M-transitive, it is also-by Lemma 1 - M'-transitive. Hence, $[R(x,y) > R(y,x)$ & $R(y,z) > R(z,y)]$ implies that $R(x,z) > R(z,x)$, viz. $P(x,z) > P(z,x)$ which - by Condition I and the fact that only j's preference over the pair (x,z) has been specified - must entail that j is decisive over the ordered pair (x,z). We thus have:

(L2.1) $\bar{D}_j(x,z)$.

Suppose, next, that $R_j(x,y)=1$, $R_j(y,x)=0$, $R_j(z,x)=1$, $R_j(x,z)=0$, $R_j(z,y)=1$, $R_j(y,z)=0$, $R_{N-\{j\}}(x,y)=0$, $R_{N-\{j\}}(y,x)=1$, $R_{N-\{j\}}(x,z)=0$, and $R_{N-\{j\}}(z,x)=1$. Then $D_j(x,y)$ implies $R(x,y) > R(y,x)$; PC requires that $R(z,x) > R(x,z)$; and M'-transitivity of R over the triple {z,x,y} implies that $R(z,y) > R(y,z)$, viz. $P(z,y) > P(y,z)$. Condition I assures as that

(L2.2) $\bar{D}_j(z,y)$.

Next, suppose that $R_j(x,y)=0$, $R_j(y,x)=1$, $R_j(y,z)=0$, $R_j(z,y)=1$, $R_j(x,z)=0$, $R_j(z,x)=1$, $R_{N-\{j\}}(x,y)=0$, $R_{N-\{j\}}(y,x)=1$, $R_{N-\{j\}}(y,z)=1$, and $R_{N-\{j\}}(z,y)=0$. Then, $\bar{D}_j(z,y)$ (already proved <u>vide</u> (L2.2)) implies $R(z,y) > R(y,z)$; PC requires $R(y,x)=1 > R(x,y)$; and M'-transitivity over {z,y,x} requires $R(z,x) > R(x,z)$; given that $P(z,x) > P(x,z)$ when only the preference of j over (x,z) has been specified, Condition I implies that

(L2.3) $\bar{D}_j(z,x)$.

Now consider the following preferences: $R_j(x,y)=0$, $R_j(y,x)=1$, $R_j(y,z)=1$, $R_j(z,y)=0$, $R_j(x,z)=1$, $R_j(z,x)=0$, $R_{N-\{j\}}(x,y)=0$,

$R_{N-\{j\}}(y,x)=1$, $R_{N-\{j\}}(x,z)=0$, and $R_{N-\{j\}}(z,x)=1$. Since $\bar{D}_j(x,z)$ (See L2.1), $R(x,z) > R(z,x)$; by PC, $R(y,x)=1 > R(x,y)$; and by M'-transitivity over the triple $\{y,x,z\}$, $P(y,z) > P(z,y)$ which, thanks to I, assure us that

(L2.4) $\bar{D}_j(y,z)$.

Let the preferences of individual j now be given by: $R_j(x,y)=0$, $R_j(y,x)=1$, $R_j(y,z)=1$, $R_j(z,y)=0$, $R_j(x,z)=0$, and $R_j(z,x)=1$. As we have already seen - from (L2.4) and (L2.3) - $D_j(y,z)$ and $D_j(z,x)$; so we must have: $R(y,z) > R(z,y)$ and $R(z,x) > R(x,z)$ whence, by M'-transitivity over $\{y,z,x\}$, $P(y,x) > P(x,y)$ which, by virtue of Condition I, leads to

(L2.5) $\bar{D}_j(y,x)$.

Suppose j's preferences are now given by: $R_j(x,y)=1$, $R_j(y,x)=0$, $R_j(y,z)=0$, $R_j(z,y)=1$, $R_j(x,z)=1$, and $R_j(z,x)=0$. Since, as already proved - see (L2.1) and (L2.2) - $\bar{D}_j(x,z)$ and $\bar{D}_j(z,y)$, we must have: $R(x,z) > R(z,x)$ and $R(z,y) > R(y,z)$ - leading, through M'-transitivity over the triple $\{z,x,y\}$ to $P(x,y) > P(y,x)$ and therefore, in view of I, to the conclusion that

(L2.6) $\bar{D}_j(x,y)$.

Now let u and v be any two distinct alternatives in X. We consider the following three mutually exclusive and completely exhaustive cases: <u>Case (i):</u> $\# [\{x,y\} \cap \{u,v\}]=2$ with, say, x=u and y=v; <u>Case (ii):</u> $\# [\{x,y\} \cap \{u,v\}]=1$ with, say, x=u; and <u>Case (iii):</u> $\# [\{x,y\} \cap \{u,v\}]=0$. Under Case (i), clearly $D_j(x,y)$ implies $\bar{D}_j(u,v)$ and $\bar{D}_j(v,u)$ (<u>vide</u> (L2.5) and (L2.6)). Under Case (ii), consider the triple $\{x,y,v\}$; again, $D_j(x,y)$ implies $\bar{D}_j(u,v)$ and $\bar{D}_j(v,u)$ (<u>vide</u> (L2.1)) and (L2.3)). Finally, consider Case (iii). First take the triple {x,y,u}. $D_j(x,y)$ implies $\overline{D}_j(x,u)$ (*vide* (L2.1)) (and therefore $D_j(x,u)$). Now take the triple {x,u,v}. Since $D_j(x,u)$ - as just proved - it must be the case (*vide* (L2.4) and (L2.2)) that $\overline{D}_j(u,v)$ and $\overline{D}_j(v,u)$. We have thus proved, as required, that if there is an individual j who is almost decisive for some x against some y in X, then j is decisive over *every* ordered pair of alternatives (u,v), viz. that j is a dictator. (Q.E.D.)

Lemma 3. Let f: $H^n_2 \to H_2$ be an FAR satisfying conditions P and I. Then, there exist $j \in N$ and $x,y \in X$ such that $D_j(x,y)$.

Proof. We shall assume the lemma to be false and derive a contradiction. Note first that, by virtue of PC, there exists an almost decisive coalition, namely the coalition N. Let C be a smallest almost decisive coalition. Let C_1 and C_2 be two mutually exclusive and completely exhaustive nonempty subsets of C, with $C_1=\{j\}$ for some $j \in N$. Let $C_3:=N-C$. Now consider the following permissible configuration of preferences over a triple of distinct alternatives $\{x,y,z\} \subseteq X$ (in what follows, α is a number in the interval (0,1)).

$P_{C_1}(x,y)=1$, $P_{C_1}(y,x)=\alpha$, $P_{C_1}(y,z)=1$, $P_{C_1}(z,y)=\alpha$, $P_{C_1}(x,z)=1$, $P_{C_1}(z,x)=\alpha$;

$P_{C_2}(x,y)=1$, $P_{C_2}(y,x)=\alpha$, $P_{C_2}(y,z)=\alpha$, $P_{C_2}(z,y)=1$, $P_{C_2}(x,z)=\alpha$, $P_{C_2}(z,x)=1$;

and

$P_{C_3}(x,y)=\alpha$, $P_{C_3}(y,x)=1$, $P_{C_3}(y,z)=1$, $P_{C_3}(z,y)=\alpha$, $P_{C_3}(x,z)=\alpha$, $P_{C_3}(z,x)=\alpha$.

Since C is an almost decisive coalition and $P_C(x,y) > P_C(y,x)$ while $P_{N-C}(x,y) > P_{N-C}(y,x)$, we must have:

(L3.1) $R(x,y) > R(y,x)$.

Suppose $R(z,y) > R(y,z)$. Then, by Condition I, C_2 must be an almost decisive coalition which - since C_2 is a strict subset of C - contradicts the fact that C is a smallest almost decisive coalition. Therefore, $\sim[P(z,y) > P(y,z)]$, viz. either

(L3.2) $P(y,z) > P(z,y)$ (i.e., $R(y,z) > R(z,y)$);

or

(L3.3) $P(y,z)=P(z,y)=0$ (recall that under constuction (2B) of the asymmetric and symmetric components of R, for all distinct $x,y \in X$: $P(x,y)=P(y,x) \rightarrow P(x,y)=P(y,x)=0$).

Suppose (L3.2) is true. Then, from (L2.1) and (L2.2) together with M'-transitivity of R over the triple $\{x,y,z\}$, one has $P(x,z) > P(z,x)$ which - in view of Condition I - would support the conclusion that C_1 is almost decisive for x against z, in contradiction of C (which is a strict superset of C_1) being a smallest almost decisive coalition. Therefore, (L3.2) must be false, and we are left with (L3.3). By PC over the pair (y,z), we must have:

(L3.4) $P(y,z) \geq \min_{i \in N} P_i(y,z) = \alpha > 0$.

(L3.3) and (L3.4) are mutually contradictory Thus, assuming the lemma to be false leads to contradiction, and we conclude that the lemma is true. (Q.E.D.).

Remark A1. It may be noted that Lemmata 2 and 3 correspond, respectively, to what Sen (1986) calls the 'Field Expansion Lemma' and the 'Group Contraction Lemma'.

Proof of Theorem 6. By Lemma 3, there exist $j \in N$ and $x, y \in N$ such that $D_j(x,y)$. By Lemma 2, j is a dictator. This completes the proof of the Theorem. (Q.E.D.).

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Figure 1

The graph of P(x,y) − P(y,x) under Construction (1) as a function of R(x,y) − R(y,x) [drawn for R(y,x) = α, Where $\alpha \in (0,1)$]