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Towards a simple interpretation of the Atkinson class of inequality indices

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ABSTRACT OF PAPER

This note presents a very simple way of interpreting the Aktinson class of inequality indices A, in 'equivalent' welfare terms, as the proportion of a cake of given size going to the poorer of two individuals in a two-person cake-sharing problem.

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Toward A Simple Interpretation of the Atkinson Class of Inequality Indices

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In this note I advance an extremely simple way of interpreting Atkinson's (1970) class of 'ethical' inequality indices. This interpretation is based on the premise that our intuitive comprehension of the notion of inequality is sharpest in the context of the canonical two-person cake-sharing problem: the share σ of the cake going to the poorer person furnishes just about the clearest idea we can have of the extent of relative inequality that obtains in any distributional exercise. The strategy therefore would be to link the value of the Atkinson inequality index A for any n-person distribution to the corresponding value of σ for a two-person distribution: this is very easily done, as will be shown in this note. But first, a quick recall of Atkinson's welfare-based approach to inequality measurement.

In everything that follows, an income distribution will be taken to be a finite-dimensional vector of incomes arranged in non-decreasing order. Let $x=(x_1,\ldots x_i,\ldots,x_n)$ be any n-person income distribution with mean $\mu(x)$. Social welfare W is of the utilitarian form:

$$(1) \quad W(x) = \sum_{i=1}^{n} u(x_i),$$

where each individual's utility function is taken to be symmetric, increasing and strictly concave, and is specialized to the constant elasticity-of-marginal-utility form:

(2) $u(x_i) = \frac{1}{\lambda}(x_i)^{\lambda}$, $\lambda\in(0,1)$[1]; $i=1,\ldots,n$.

Given a distribution x, Atkinson defines the <u>equally distributed equivalent income</u> $\hat{\mu}(x)$ to be that level of income such that if it is equally shared, the resulting level of welfare is the same as that which obtains under the distribution x:

(3) $W(\hat{\mu}(x),\ldots,\hat{\mu}(x)) = W(x_1,\ldots,x_n)$.

Given (1) - (3), we have

(4) $\hat{\mu}(x) = [\frac{1}{n}\sum_{i=1}^{n}(x_i)^{\lambda}]^{1/\lambda}$.

Consider the inequality index $\tilde{A}$ which is obtained as the proportionate difference between $\mu(x)$ and $\hat{\mu}(x)$:

(5) $\tilde{A}(x) = 1-\hat{\mu}(x)/\mu(x)$.

$\tilde{A}$ is the conventional way in which the Atkinson index is presented, but in this form the index is not appropriately normalized to yield a maximum value of unity for all values of n. This problem can be circumvented in the following fashion. Let $\underline{\hat{\mu}}(x)$ be the minimum possible value of the equally distributed equivalent income; it is that level of income which, if it is equally shared, will yield the same level of welfare as is yielded by a redistribution of incomes in x such that just one person appropriates the entire income:

(6) $W(\underline{\hat{\mu}}(x),\ldots,\underline{\hat{\mu}}(x)) = W(0,0,\ldots,n\mu(x))$.

In view of (1), (2) and (6), we have:

(7) $\underline{\hat{\mu}}(x) = n^{\frac{\lambda-1}{\lambda}}\mu(x)$.

A properly normalized version of the index $\tilde{A}$, which will ensure that its maximum value is always exactly unity, irrespective of the dimensionality of the income vector, is given by

$$(8) \quad A(x) = \frac{\mu(x) - \hat{\mu}(x)}{\mu(x) - \hat{\underline{\mu}}(x)},$$

which, using (5) and (7), yields:

$$(9) \quad A(x) = \left[\frac{1}{1-n^{\frac{\lambda-1}{\lambda}}} \right] \tilde{A}(x)^{2>}.$$

In what follows, by the 'Atkinson index of inequality' I shall mean the expression on the right hand side of (8).

Now, given an n-person distribution x and some value of the 'inequality-aversion' parameter λ, it is not always easy to conceptualize precisely what a particular value of the inequality index A(x) 'really means' in terms of categories of inequality that we may be familiar with at a more 'primitive' level. To facilitate such an understanding, consider the following.

Given an n-person distribution x, define a 'dichotomously allocated equivalent distribution' to be an ordered two-person income vector $x^*=(x_1^*, x_2^*)$ such that x^* and x have both the same means and the same values of the Atkinson inequality index. Then, $\mu(x^*)=\mu(x)$ implies

$$(10) \quad x_1^* + x_2^* = 2\mu(x);$$

and $A(x^*)=A(x)$ - making appropriate use of (4),(7) and (8) - implies

$$(11)\quad \frac{\mu(x) - \left[\frac{1}{2} \left\{ (x_1^*)^\lambda + (x_2^*)^\lambda \right\} \right]^{1/\lambda}}{\left(1-2^{\frac{\lambda-1}{\lambda}}\right)\mu(x)} = A(x).$$

Substituting for x_2^* from (10) into (11), and after suitable manipulation of (11), we obtain:

$$(12)\quad (x_1^*)^\lambda + (2\mu-x_1^*)^\lambda = 2[\mu(x)\{1-A(x)(1-2^{\frac{\lambda-1}{\lambda}})\}]^{1/\lambda}.$$

Designating the poorer person's income-share in the distribution x^* by σ_A, we have: $\sigma_A = x_1^* / 2\mu(x)$, whence $x_1^* = 2\mu(x)\sigma_A$; substituting for x_1^* into (12) and simplifying, yields:

$$(13)\quad (\sigma_A)^\lambda + (1-\sigma_A)^\lambda = 2^{1-\lambda}[1-A(x)(1-2^{\frac{\lambda-1}{\lambda}})]^\lambda.$$

Using (13) we can solve for σ_A, given any value of A (though a closed-form solution expressing σ_A as a function of A is not available). Notice from (13) that when $A(x)=0$ (no inequality), $\sigma_A=\frac{1}{2}$ (equal share), and when $A(x)=1$ (perfect concentration), $\sigma_A=0$ (the poorer person receives nothing). In general, given the value of the index A for any n-person distribution, one can transform it into an 'equivalent' value of σ_A - the share of the poorer person in a two-person distribution - which affords an immediate and vivid picture of the extent of inequality that A 'signifies'.

Exactly the same line of reasoning as above can be employed to interpret the Gini coefficient of inequality. Employing a social welfare function which is simply a weighted sum of individual incomes, the weight on any income being its rank-order in the income vector [3], we have:

$$(14)\quad W(x) = \sum_{i=1}^{n} (n+1-i)x_i.$$

Using (3), (6), (8) and the welfare function specified in (14), it is easy to check that the Gini coefficient of inequality - which is simply the Atkinson index for the welfare function specified in (14) - is given by

$$(15)\quad G(x) = \frac{n+1}{n-1} - \frac{2}{n(n-1)\mu} \Sigma(n+1-i)x_i.$$

Exploiting the definition of a 'dichotomously allocated equivalent distribution' it is a routine matter to derive the functional relation between the share σ_G going to the poorer of the two persons in the distribution x^*, and $G(x)$, as

$$(16)\quad \sigma_G = \frac{1}{2}(1-G(x)).$$

A further matter of some interest in this connection has to do with Sen's (1976) index of 'real national income' - call it R - which is given by the quantity $\mu(1-G)$. Recalling that $\sigma_G \equiv x_1^*/2\mu(x)$, it is immediate, given (16), that

$$(17)\quad x_1^* = \mu(x)\ [1-G(x)] \equiv R(x).$$

It turns out, therefore, that for any society represented by the income vector x for which the mean is $\mu(x)$ and the Gini coefficient of inequality is $G(x)$, Sen's index of real national income $R(x)$ can be identified simply with the income level x_1^* of the poorer of the two persons in the 'dichotomously allocated equivalent distribution' x^*: this is, of course, entirely in consonance with the 'Rawlsian' prescription of indentifying the welfare of a society with the prospect confronting its worst-off member.

Finally, by way of an empirical illustration, I present in Table 1 the values of σ_A and σ_G for corresponding values of A and G that have been computed from 1970 income-distribution data for Malaysia by Anand (1983).

[Table 1 to be inserted here]

From Table 1 we can see, for example, that when $\lambda=0.5$, the welfare considerations underlying the Atkinson index imply that the extent of interpersonal inequality that obtains in the distribution of incomes in Malaysia is 'comparable' to a situation in which the poorer of two persons receives just a little under a fifth of a cake that has to be divided two ways (this share is just a little over a quarter for the Gini coefficient): this 'equivalence translation' affords a graphic way of comprehending inequality.

Table 1: The Atkinson Inequality Index A, the Gini Coefficient of Inequality G, and the Inequality Indices σ_A and σ_G for the Distribution of Individuals by Per Capita Household Income: Malaysia, 1970.

λ	$\tilde{A}$	A	σ_Ax100 percent	G	σ_Gx100 percent
0.75	0.1162	0.1185	28.34	--	--
0.50	0.2124	0.2124	19.19	--	--
0.25	0.3026	0.3026	17.40	--	--
0.10	0.3807	0.3807	9.32	--	--
0.01	0.7615	0.7615	1.37	--	--
--	--	--	--	0.4980	25.10

Note: (1) This table is based on data furnished in Tables 3-8 and 3-9 of Anand (1983).

(2) Anand's computations are based on data in the Malaysian Post-Enumeration Survey of 1970: the sample size of the survey (number of individuals) is reported to be 134,186.

(3) Anand's reported values of $\tilde{A}$ (Table 3-9 in Anand, 1983) have been mapped into their corresponding values of A by using the relation

$$A = \left[\frac{1}{1-n^{\frac{\lambda-1}{\lambda}}}\right]\tilde{A}$$

(See Equation (9) in the text), where n=134,186. Except for λ=0.75, there is no discrepancy between A and $\tilde{A}$ up to the fourth decimal point.

(4) The value of G reported in the last row of the table is from Table 3.8 of Anand (1983).

(5) σ_A and σ_G has been computed by employing equations (13) and (16) respectively in the text.

NOTES

1> Non-positive values of λ are not considered, because of the problems occasioned by these in the presence of zero-incomes in a distribution: for a discussion of these difficulties, see Anand (1983; pages 84-86).

2> Equation (9) suggests, to put matters loosely, that $\tilde{A}$ is a 'valid' approximation of A for 'large' n. This is a somewhat misleading inference, and for a more precise statement, consider the following. Let $\varepsilon \equiv (1-\tilde{A}/A)$ measure the 'margin of discrepancy' between A and $\tilde{A}$, and let ε^* be that value of ε which we are prepared to tolerate. Then, given ε^*, we can say that for every $\lambda \in (0,1)$ there exists a sufficiently large n - call it n^* - for which $\varepsilon = \varepsilon^*$; using (9), it is easy to check that the required value of n is given by $n^* = (\varepsilon^*)^{\lambda/(\lambda-1)}$. Thus, for example, if ε^* is pegged at 0.01 (meaning that we are prepared to admit a 1 per cent margin of discrepancy between A and $\tilde{A}$), then for $\lambda=0.5$ it can be verified that n^* must be 100; while for $\lambda=0.9$, n^* must be 10^{18}! In principle, therefore, given that the Atkinson index has always been interpreted as the index $\tilde{A}$, one should be a little wary of treating other shcolars' economy-wide estimates of $\tilde{A}$ as 'valid' approximations of A, if it is the latter that one is after. (In this connection, see Note (3) to Table 1 in the text).

3> See Sen (1973).

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